3.0 Position, Velocity, Acceleration and Force Analysis of Linkages

Text Reference:

Students should review Chapters 3 (Displacement and Velocity Analysis), Chapter 4 (Acceleration Analysis), and Chapter 5 (Introduction to Dynamics of Mechanisms). More specific references are presented at the beginning of each section.

All dynamic analysis commences with Newton’s second law:

\[ \vec{F} = m \ddot{a} \]

Most of our effort will be to obtain suitable expressions for \( \ddot{a} \), the acceleration of points on the linkage (usually joints or centres of gravity). To do this we require velocity and position analyses.

3.1 Position Analysis

Position analysis is the first step in any kinematic analysis of a linkage, since velocity, acceleration and force analyses require the present position to establish the direction of the various vectors. We will concentrate on the analysis of four-bar linkages to present general displacement analyses. Figure 3.1 illustrates a typical 4-bar linkage.

![Four-bar linkage diagram](image)

Figure 3.1: Four-bar linkage

In this case, we have labeled the links \( s, l, p, \) and \( q \), where \( s \) corresponds to the shortest link, \( l \) corresponds to the longest link, and \( p \) and \( q \) correspond to the intermediate length links. This nomenclature is chosen to facilitate a simple screening analysis of the linkage to determine what type of motion is possible. This screening analysis is Grashof’s criteria.
3.1.1 *Grashof’s Criteria*

*Text Reference:* Grashof’s criteria is discussed in Section 3.1.

Grashof’s criteria is used to determine whether or not at least one of the links can rotate $360^\circ$. It states that:

> the sum of the shortest and longest links of a planar four-bar mechanism cannot be greater than the sum of the remaining two links if there is to be continuous relative rotation between the two links.

This can be demonstrated with the help of Figure 3.2, where the short and long links have been placed adjacent to one another.

![Grashof Mechanism](image)

For the figure on the left, the Grashof condition is satisfied, $s + l < p + q$, and the shortest link can rotate $360^\circ$. At the position indicated, the short and long link are aligned, but since the sum of the lengths of these two links are less than the sum of the other links, the short link can continue to rotate by bringing the output link, $q$, back. On the right, Grashof’s condition is not satisfied. Therefore, as the short link is moved around, the fact that $s + l > p + q$ means that the short link cannot complete its revolution.

This situation can be generalized by recognizing that in any inversion of this linkage, the shortest can still rotate relative to the rest of the mechanism if Grashof’s condition is satisfied. In addition, if the longest link is opposite the shortest link, we are still assured of having $360^\circ$ rotation of the short link if Grashof’s condition is satisfied.

If Grashof’s condition is satisfied, $s + l < p + q$, we have four possible types of Grashof mechanisms, depending on which link is fixed:

- A *crank-rocker* mechanism is obtained if the shortest link is adjacent to the ground link. The shortest link becomes the crank, which can rotate $360^\circ$. The output link can only ‘rock’ back and forth, Figure 3.3 (left).
If the shortest link is the follower, then the input can only rock back and forth, and the output rotates through $360^\circ$, forming a rocker-crank, Figure 3.3 (right).

If the shortest link is the frame, then a drag-link mechanism is formed, Figure 3.4 (left). Both input and output links can rotate through $360^\circ$.

If the shortest link is opposite the frame, then a double rocker mechanism results, Figure 3.4 (right). The coupler (shortest link) rotates through $360^\circ$.

If Grashof’s criteria is not satisfied, and $s + l > p + q$, then we have four possible triple-rockers (like the double-rocker in Figure 3.4 (right), except that the coupler also cannot rotate through $360^\circ$), depending on which link is the frame.

If $s + l = p + q$, we still can have the motion of the Grashof mechanisms illustrated in Figures 3.3 and 3.4, but each will suffer from a \textit{change-point condition}. That is, the
centre lines of all links will become collinear, and we cannot be certain (without considering the dynamics) which position the linkage will take up, Figure 3.5.

![Figure 3.5: Example change-point mechanism approaching the change-point.](image)

A position analysis is used to determine the position of all other links given the position of the input link, the lengths of all other links and the ‘topography’ (characteristics of the links and joints) of the linkage. There are several methods by which we can perform a position analysis: graphical, vector and analytical.

One of the objectives of a position analysis is to determine the transmission angle, $\gamma$, the angle between the coupler link and the output link. This is illustrated in Figure 3.6. It is best if this angle lies in the range $45^\circ < \gamma < 135^\circ$ for an effective transfer of motion between the input and output.

![Figure 3.6: Definition of transmission angle.](image)

### 3.1.2 Vector Position Analysis

**Text Reference:** Graphical displacement analysis is covered in section 3.2. Don’t try to understand his ‘analytical method’, section 3.3.

The general approach to the analysis of the position of a linkage is to represent the links of the planar linkage by two-dimensional vectors and ‘close-the-loop’ by expressing the position of one of the joints in two different ways, and equating the result. Consider the four-bar linkage in Figure 3.7.
Assuming that the origin of the vector system is at $O_2$, we can express the position of joint $B$ in two different ways, and equate them to obtain the following vector equation:

$$\vec{R}_A + \vec{R}_{B/A} = \vec{R}_{O_4/O_2} + \vec{R}_B$$

Consider the situation in which we are given the length of all links and the configuration of the linkage (joint locations and types). Our task is then to determine the position of the coupler and output links given the position of the input link. Therefore, we know the vectors $\vec{R}_A$ and $\vec{R}_{O_4/O_2}$ explicitly. In addition, we know the magnitudes of $\vec{R}_{B/A}$ and $\vec{R}_B$, since they are given by the lengths of the coupler and output links, respectively. The two unknowns in this system of equations are then the angles of these two vectors (links).

There are several techniques available to solve this equation, based on different ways of expressing the vectors: graphical, $x$-$y$ components, and complex numbers.

**Graphical Position Analysis**

The simplest way to perform this displacement analysis is by using geometrical constructions to solve the above vector equation. For the pin-jointed four-bar linkage in Figure 3.7, we are given the position of the input link and the lengths of the output and coupler links. We then proceed to make a graphical construction as illustrated in Figure 3.8.
Figure 3.8: Graphical position analysis for a four-bar linkage.

The procedure is to draw the input and ground links (or vectors), which are known (the angular orientation of the input link is expressed with respect to ground). Then, draw circles equal to the known radii of the coupler and output links centred on the end of the input and ground links, respectively. The intersection of the circles represents the position of the joint connecting the coupler and output links. The angles of the links can then be measured directly from the drawing using a protractor. Note that this construction can also be made using a drawing package such as AUTOCAD.

One advantage of this graphical technique is that it is clear from the figure that there are two possible configurations for this linkage. We must use additional information (such as a prior position) to decide between the two possible configurations.

Example 3.1: In a small four-bar linkage, the ground link is 100 mm, the input link is 25 mm, the coupler is 75 mm and the output link is 100 mm. Determine what type of linkage this is, and its position for an input angle, \( \theta_2 \), of 45°, as measured counter clockwise from the ground link. Use a graphical construction.

Vector Component Position Analysis

In this approach, the vector equation above is expressed in \( x \) and \( y \) components, using the angles of each link with respect to ground (\( \theta_1 = 0 \)), Figure 3.9. Terms corresponding to these components are then separated to form two simultaneous algebraic equations.
Using the conventions illustrated in the inset to Figure 3.9, we arrive at the following
vector equation:

\[ R_2 \left( \cos \theta_2 \hat{i} + \sin \theta_2 \hat{j} \right) + R_3 \left( \cos \theta_3 \hat{i} + \sin \theta_3 \hat{j} \right) = R_1 \hat{i} + R_4 \left( \cos \theta_4 \hat{i} + \sin \theta_4 \hat{j} \right) \]

Which gives the following simultaneous algebraic equations for the \( x \) and \( y \) directions:

\[
R_2 \cos \theta_2 + R_4 \cos \theta_3 = R_1 + R_4 \cos \theta_4 \\
R_2 \sin \theta_2 + R_4 \sin \theta_3 = R_4 \sin \theta_4
\]

In these equations, all the radii are known, as well as \( \theta_2 \), leaving two equations in two
unknowns: \( \theta_3 \) and \( \theta_4 \). Unfortunately, these are not linear equations (they are in fact
transcendental equations), and special iterative techniques are required to solve them.
These require an initial guess at the solution, which is subsequently refined. Special care
should be taken to ensure that both solutions are found. In a case where a simulation is
being performed, the position at the previous time-step can be used as the initial guess.
The actual solution of these equations is best done numerically with a commercial
package such as MATHCAD.

**Complex Number Vector Analysis for Position**

Vectors in two-dimensions can also be represented as complex numbers. If we replace
the vector representation illustrated in Figure 3.9 with the complex representation
illustrated in Figure 3.10, we can solve the same example.
In complex number notation, and taking advantage of Euler’s equation relating the component form to the exponential form for complex numbers, we can express the vector as:

\[ \vec{R} = R e^{i\theta} = R(\cos\theta + i\sin\theta) \]

In practice, unless we use a computer program which handles complex numbers directly, this uses exactly the same approach as the vector analysis above. This complex number format is also convenient when we differentiate vector equations to obtain relations for velocity and acceleration.

In any case, since we are generally trying to solve for angles, we are left with two simultaneous transcendental equations, for the real and imaginary components, which must be solved. For simple problems, such as four-bar linkages, it is often easier to solve for the position using analytical geometry.

### 3.1.3 Analytical Geometry Position Analysis

To determine the position of a four-bar linkage, Figure 3.11, using analytical geometry, we draw a diagonal between points \( A \) and \( O_2 \), which has length \( l \). This forms two triangles, one of which includes the known input angle \( \theta_2 \).
The first step in this analysis is to determine the length, \( l \), of the diagonal. This is done using the cosine law:

\[
l^2 = a^2 + b^2 - 2ab \cos \theta
\]

where \( a \) and \( b \) are the lengths of the other two sides of the triangle, and \( \theta \) is the angle opposite to side of length \( l \). Notice that this reverts to the Pythagorean formula when \( \theta \) is \( 90^\circ \) and \( l \) becomes the hypotenuse of a right triangle. Once we have determined \( l \), we know all the lengths of the two triangles, and we can determine all of the remaining angles through repeated application of the cosine law.

Note that this solution permits a second solution, whereby the ‘upper’ triangle is below the diagonal, \( l \).

Example 3.2: Solve Example 3.1 using analytic geometry.

### 3.1.4 Limiting Positions for Linkages

One of the most common applications of displacement analysis for linkages is to determine the limiting position of the links. Consider the crank-rocker mechanism in Figure 3.12.

![Figure 3.12: Example crank-rocker four-bar linkage, with limiting positions.](image)

We generally wish to know the range of motion for the output link. This can be determined by doing a complete position analysis for all possible input angles. However, by identifying the type of linkage first, a crank-rocker in this case, and making certain observations about the motion, we can solve for the limits directly.

The output link will reach its extreme positions when the input and coupler links are collinear. Thus, the linkage will form a triangle in its extreme positions, Figure 3.12,
with the length of one side either the sum of the lengths of the input and coupler links, or the difference. One can then apply any of the above position analysis techniques to determine the extents of the output link.

Example 3.3: Determine the extreme positions for the output link for the linkage in Example 3.1, using both graphical and analytical techniques.